Products and slices.

August 10, 2017 11:19 AM

Thm: (Mastrand's product Thm).

$A \subset \mathbb{R}^d$, $B \subset \mathbb{R}^n$ - compact. Then:

$$\mathrm{Hdim}\, A + \mathrm{Hdim}\, B \le \mathrm{Hdim}(A \times B) \le \mathrm{Hdim} A + \mathrm{Pdim}\, B$$

Pf

LHS: Pick $\alpha < \mathrm{Hdim}\, A$, $\beta < \mathrm{Hdim}\, B$,

By Frostman's Lemma, $\exists\, \mu$ on A, ν on B:

$\mu(B(x,r)) \le C_d r^\alpha$

$\nu(B(y,r)) \le C_n r^\beta$

Then $\mathrm{supp}(\mu \times \nu) \subset A \times B$.

$(\mu \times \nu)(B((x,y),r)) \le \mu(B(x,r))\, \nu(B(y,r)) \le C_d C_n r^{\alpha+\beta}$.

By Mass Distribution Principle, $\mathrm{Hdim}(A \times B) \ge \alpha + \beta$.

RHS: First let us prove it for $\overline{\mathrm{Mdim}}\, B$ instead of $\mathrm{Pdim}\, B$.

Then we can cover $B = \cup B_i$, with $\sup \overline{\mathrm{Mdim}}\, B_i \le \mathrm{Pdim}\, B + \varepsilon$,

and use $\mathrm{Hdim}\, A \times B = \sup \mathrm{Hdim}(A \times B_i)$.

Choose $\alpha > \mathrm{Hdim}\, A$, $\beta > \overline{\mathrm{Mdim}}\, B$.

Then, $\exists\, \varepsilon_0$: $\varepsilon < \varepsilon_0$, $N(\varepsilon, B) \le \varepsilon^{-\beta}$.

$\forall \delta > 0$ $\exists$ $A \subset \cup A_i$, $\mathrm{diam}\, A_i < \varepsilon_0$, $\sum (\mathrm{diam}\, A_i)^\alpha < \delta$.

For each i, $\cancel{\exists}$ covering $B = \bigcup_{k=1}^{L_i} B_{ik}$, $\mathrm{diam}\, B_{ik} = \mathrm{diam}\, A_i$.

Then $L_i \le (\mathrm{diam}\, A_i)^{-\beta}$.

We see that $\{A_i \times B_{ik}\}$ cover $A \times B$,

$$\sum_j \sum_{k=1}^{L_j} (\mathrm{diam}\, A_j \times B_{jk})^{\alpha+\beta} \le \sqrt{2} \sum_j L_j (\mathrm{diam}\, A_j)^{\alpha+\beta} \le \sqrt{2} \sum (\mathrm{diam}\, A_j)^\alpha < \sqrt{2}\,\delta$$

Examples: 1) C - usual Cantor set

$\mathrm{Hdim}(C \times C) = 2 \log_3 2$ (both sides of the inequality are equal).

2) Lacunary sets.

$S = \bigcup_{j=1}^{\infty} [(2j-1)!, (2j)!]$,

$S^c = \bigcup_{j=1} [(2j)!, (2j+1)!]$.

$\underline{\mathrm{Mdim}}\, A_S = \underline{\mathrm{Mdim}}\, A_{S^c} = \underline{d}(S) = \underline{d}(S^c) = 0$, so $\mathrm{Hdim} = 0$ also, but
(lower density)

$\overline{\mathrm{Mdim}}\, A_S = \mathrm{Pdim}\, A_S = 1$, (as we saw)

Same for A_{S^c}.

Th $E = A_S \times A_{S^c}$, then the map

$f: (x,y) \to (x+y)$ is Lipshitz,

$f(E) \supset [0,1]$ (we can expand any number,

so $\mathrm{Hdim}\, E \ge 1 > \mathrm{Hdim} A_S + \mathrm{Hdim} A_{S^c}$.

But $\mathrm{Hdim}\, E \le \mathrm{Hdim}\, A_S + \mathrm{Pdim}\, A_{S^c} = 1$, so $\mathrm{Hdim}\, E = 1$ ∎

3) $A = [0,1]$, $B = A_S$.

Then

$1 \le \mathrm{Hdim}(A \times B) \le \mathrm{Hdim}\, B + \mathrm{Pdim}\, A = 1 < \mathrm{Hdim}[0,1] + \mathrm{Pdim}\, B = 2$ ∎

Let us now turn to slices:

Thm (Mastrand Slicing Theorem). $\mathrm{Hdim}\, A > 2$

Let $A \subset \mathbb{R}^n$ be a Borel set ... be an ... n-dimensional

Let us now turn to slices:

Thm (Mastrand Slicing Theorem). $\operatorname{Hdim} A > \alpha$

Let $A \subset \mathbb{R}^n$ be a Borel set, E_m be an m dimensional subspace, and μ be a measure on $E_m^\perp$ with $\mu(B(x,r)) \le C r^\alpha$ (α-smooth)

Then μ-a.e. $x \in E_m^\perp$,

$$\operatorname{Hdim}(A \cap (E_m + x)) \le \operatorname{Hdim} A - \alpha.$$

Corollary. Lebesgue a.e. on $E_m^\perp$ (or on $\mathbb{R}^n$),

$$\operatorname{Hdim}(A \cap (E_m + x)) \le \operatorname{Hdim} A - (n-m).$$

Pf of Thm.

Take $\gamma > \operatorname{Hdim} A$. Want to show:

$\int m_{\gamma-\alpha}(A \cap (E_m + x))\, d\mu(x) < \infty$. This would imply that μ-a.e.

$m_{\gamma-\alpha}(A \cap (E_m + x)) < \infty$, so $\operatorname{Hdim}(A \cap (E_m + x)) \le \gamma - \alpha$, μ-a.e. Take $\gamma_n \downarrow$.

By rotation, can make $E_m = \mathbb{R}^m$, $E_m^\perp = \mathbb{R}^{n-m}$,

Cover A by cubes $\{Q_i = A_i \times B_i,\ A_i \subset \mathbb{R}^{n-m}, B_i \subset \mathbb{R}^m\}$,

$\operatorname{diam} Q_i < \varepsilon$, $\sum (\operatorname{diam} Q_i)^\gamma < \varepsilon$.

For $y \in \mathbb{R}^m$, $x \in \mathbb{R}^{n-m}$, write

$$f(x,y) := \sum (\operatorname{diam} Q_i)^{\gamma - m - \alpha} \chi_{Q_i}(x,y).$$

Note that

$$\int_{\mathbb{R}^{n-m}} \int_{\mathbb{R}^m} f(x,y)\, dy\, d\mu(x) \le \sum (\operatorname{diam} Q_i)^{\gamma - m - \alpha} (\operatorname{diam} B_i)^m \mu(A_i) \le C \sum (\operatorname{diam} Q_i)^\gamma < C\varepsilon.$$

Define

$$Q_i^x = \begin{cases} B_i, & x \in A_i \\ \emptyset, & x \notin A_i \end{cases}$$

$A \cap \{x + E_m\} \subset \bigcup Q_i^x$, so, by Fubini,

$$\iint f(x,y)\, dy\, d\mu(x) = \int \left(\int f(x,y)\, dy \right) d\mu(x) \ge$$

$$\int \sum |\operatorname{diam} Q_i^x|^{\gamma - \alpha}\, d\mu(x) \ge$$

$$\int m^{\varepsilon}_{\gamma-\alpha}(A \cap (E_m + x))\, d\mu(x).$$

Let $\varepsilon \to 0$ to get

$$\int m_{\gamma-\alpha}(A \cap (E_m + x))\, d\mu(x) = 0$$